Indian Statistical Institute
Midterm Examination 2015-2016
M.Math First Year
Functional Analysis

Time : 3 Hours Date : 26.02.2016 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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Q1. (10 marks) Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Prove that for each $x \in \mathcal{H}$

$$\lim_{m\to\infty} \langle x, e_m \rangle = 0.$$

Q2. (15 marks) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be a pair of equivalent norms on a linear space X. Prove that $\|\cdot\|_1$ and $\|\cdot\|_2$ yields equivalent norms on the set of bounded linear operators on X.

Q3. (15 marks) Let X and Y be two Banach spaces over $\mathbb{C}$ and $\varphi : X \times Y \to \mathbb{C}$ be a bilinear map and continuous in each variable separately. Prove that φ is jointly continuous.

Q4. (15 marks) Let $\mathcal{F}$ be a proper finite dimensional subspace of a normed linear space X. Prove that there exists an unit vector $x \in X$ such that

$$\|x - y\| \geq 1 \qquad (\forall y \in \mathcal{F}).$$

Q5. (15 marks) Prove or disprove (with justification): $T : c_{00} \to \mathbb{C}$ defined by $T(\{a_n\}) = \sum_n a_n$ is a bounded linear functional (w.r.t. usual sup norm).

Q6. (15 marks) Let T be a linear map on a Hilbert space $\mathcal{H}$ and

$$\langle Tx, y \rangle = \langle x, Ty \rangle,$$

for all $x, y \in \mathcal{H}$. Prove that T is bounded.

Q7. (15 marks) Let $\mathcal{S}$ be a subspace of a Hilbert space $\mathcal{H}$. Prove that

$$(\mathcal{S}^{\perp})^{\perp} = \overline{\mathcal{S}}.$$

Q8. (15 marks) Let A be a σ-finite measure space and $\varphi \in L^{\infty}(A)$. Prove that the multiplication operator M_{φ} has bounded inverse if and only if there exists $c > 0$ such that

$$|\varphi(x)| \geq c \qquad (\forall x \in A \ a.e).$$